Advanced Mathematical Models & Applications
Vol.9, No.3, 2024, pp.431-436
https://doi.org/10.62476/amma93431

SPECTRAL PROPERTIES OF AN IMPULSIVE STURM–LIOUVILLE OPERATOR WITH COMPLEX PERIODIC COEFFICIENTS

R.F. Efendiev*, **M.R. Sharifli**

Baku Engineering University, Mathematics Department, Baku, Azerbaijan

Abstract. This study focuses on exploring the spectral properties and scattering function of the impulsive operator generated by the Sturm–Liouville equation with complex periodic potentials. We introduce a new approach to investigate the spectral singularities and eigenvalues of the operator. Additionally, we establish the finiteness of eigenvalues and spectral singularities with finite multiplicities under specific conditions.

Keywords: Impulsive operators, Sturm–Liouville equations, periodic potential, Spectral singularities.
AMS Subject Classification: Th34B37; 34L05; 34L25; 34L40; 34B09.
Corresponding author: Rakib, Efendiev, Baku Engineering University, H.Aliyev,120, Baku, Azerbaijan, Tel.: +994502122834, e-mail: *refendiyev@beu.edu.az*
Received: 20 June 2024; Revised: 17 August 2024; Accepted: 12 September 2024;
Published: 4 December 2024.

1 Introduction

Consider the Sturm–Liouville equation on the semi-axis

$$\begin{gathered} -y'' + q(x)\,y = \lambda^2 \rho(x)\,y, \\ x \in [0, x_0) \cup (x_0, \infty) \end{gathered} \tag{1}$$

with boundary condition

$$y(0) = 0 \tag{2}$$

where λ is a spectral parameter,

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \quad \sum_{n=1}^{\infty} |q_n| < \infty, \tag{3}$$

the density function $\rho(x)$ determined as

$$\rho(x) = \begin{cases} 1, & 0 \le x < x_0 \\ \beta^2, & x_0 < x < \infty \end{cases} \; ; \quad \beta \neq 1, -1 \tag{4}$$

and for complex numbers $\alpha_i, \;\; i = 1, 2, 3, 4$ fulfilled impulse condition

$$\begin{bmatrix} y(x_0^+) \\ y'(x_0^+) \end{bmatrix} = B \begin{bmatrix} y(x_0^-) \\ y'(x_0^-) \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \quad \det B \neq 0 \tag{5}$$

How to cite (APA): Efendiev, R.F., Sharifli, M.R. (2024). Spectral properties of an impulsive Sturm–Liouville operator with complex periodic coefficients. *Advanced Mathematical Models & Applications*, *9*(3), 431-436 https://doi.org/10.62476/amma93431

In various physical applications, boundary value problems with jump conditions arise due to the discontinuous properties of materials (Bairamov et al., 2018; Hald, 1984; Efendiev et al., 2016, 2022; Aghamaliyeva et al., 2023).

The paper deals with the theory of inverse spectral problems. Such problems consist of recovering operators based on their spectral characteristics (Efendiev et al., 2021, 2022; Gasimov, 2022). The most complete results of inverse problem theory have been obtained for the simplest second-order Sturm-Liouville operators in the case of $\rho(x) = 1$. For that case, the spectral analysis of the problem having discrete and continuous spectra was begun by Naimark (1954). Naimark proved that some poles of the resolvent kernel are not eigenvalues of the operator. Additionally, he showed that these poles referred to as spectral singularities by Schwartz (Naimark, 1954), pose a mathematical obstacle to the completeness of the eigenvectors and are integrated within the continuous spectrum.

In this work, we are interested in the impulsive Sturm-Liouville operator with complex periodic potentials on the semi-axis. The operator's spectrum is analysed using the density function and impulsive condition.

Determining a transfer matrix is difficult, but obtaining some spectral information through it is possible.

2 Representation of Fundamental Solutions

Our main goal here is on the solutions to the main equation

$$-y'' + q(x)\,y = \lambda^2 \rho(x)\,y,$$
$$x \in [0, x_0) \cup (\,x_0, \infty)$$

that will be needed later.

Theorem 1. *Let $q(x)$ be of the form (4) and $\rho(x)$satisfy condition (3). Then for the given $x \in (x_0, \infty)$, $\lambda \in \overline{C_+} = \{\lambda \in C; \operatorname{Im}\lambda > 0\}$ equation (1) has special solutions of the form*

$$f(x,\lambda) = e^{i\lambda\beta x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2\lambda\beta} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right)$$

where the numbers $V_{n\alpha}$ are determined from the recurrent relations.

The proof of the theorem is similar to that of Efendiev et al. (2023) and therefore we do not cite it here.

It is well known that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the fundamental solutions of (1) in the interval $x \in [0, x_0)$, satisfying the given conditions

$$\varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1,$$
$$\psi(0,\lambda) = 1, \quad \psi'(0,\lambda) = 0,$$

respectively.

On the other hand, (1) admits another solution for $x \in (x_0, \infty)$, $\lambda \in \overline{C_+} = \{\lambda \in C; \operatorname{Im}\lambda < 0\}$

$$f(x,-\lambda) = e^{-i\lambda\beta x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n-2\lambda\beta} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right)$$

By analogy to Efendiev et al. (2021), it is easy to see that equation (1) has fundamental solutions $f(x,\lambda)$,$f(x,-\lambda)$ for which

$$W[f(x,\lambda), f(x,-\lambda)] = -2i\lambda\beta$$

where $W[f,g] = fg' - f'g$ is satisfied.

Then each solution of equation (1) may be represented as a linear combination of these solutions which makes true the following lemma

Lemma 1. *Each solution of equation (1) may be represented as a linear combination of these solutions*

$$\begin{cases} y_-(x,\lambda) = A_-(\lambda)\psi(x,\lambda) + B_-(\lambda)\varphi(x,\lambda), & 0 \le x < x_0 \\ y_+(x,\lambda) = A_+(\lambda) f(x,\lambda) + B_+(\lambda) f(x,-\lambda), & x_0 < x < \infty \end{cases}$$

Here $y_+(x,\lambda)$ and $y_-(x,\lambda)$ are solutions of equation(1) on $[0,x_0)$ and (x_0,∞) respectively. Then, from impulse condition (5) we have

$$\begin{bmatrix} A_+(\lambda) f\left(x_0^+,\lambda\right) + B_+(\lambda) f\left(x_0^+,-\lambda\right) \\ A_+(\lambda) f'\left(x_0^+,\lambda\right) + B_+(\lambda) f'\left(x_0^+,-\lambda\right) \end{bmatrix} = B \begin{bmatrix} A_-(\lambda)\psi\left(x_0^-,\lambda\right) + B_-(\lambda)\varphi\left(x_0^-,\lambda\right) \\ A_-(\lambda)\psi'\left(x_0^-,\lambda\right) + B_-(\lambda)\varphi'\left(x_0^-,\lambda\right) \end{bmatrix}$$

By taking into account that

$$\begin{bmatrix} A_+(\lambda) f\left(x_0^+,\lambda\right) + B_+(\lambda) f\left(x_0^+,-\lambda\right) \\ A_+(\lambda) f'\left(x_0^+,\lambda\right) + B_+(\lambda) f'\left(x_0^+,-\lambda\right) \end{bmatrix} = \begin{bmatrix} f\left(x_0^+,\lambda\right) & f\left(x_0^+,-\lambda\right) \\ f'\left(x_0^+,\lambda\right) & f'\left(x_0^+,-\lambda\right) \end{bmatrix} \begin{bmatrix} A_+(\lambda) \\ B_+(\lambda) \end{bmatrix}$$

and analogously

$$\begin{bmatrix} A_-(\lambda)\psi\left(x_0^-,\lambda\right) + B_-(\lambda)\varphi\left(x_0^-,\lambda\right) \\ A_-(\lambda)\psi'\left(x_0^-,\lambda\right) + B_-(\lambda)\varphi'\left(x_0^-,\lambda\right) \end{bmatrix} = \begin{bmatrix} \psi\left(x_0^-,\lambda\right) & \varphi\left(x_0^-,-\lambda\right) \\ \psi'\left(x_0^-,\lambda\right) & \varphi'\left(x_0^-,-\lambda\right) \end{bmatrix} \begin{bmatrix} A_-(\lambda) \\ B_-(\lambda) \end{bmatrix}$$

easily we obtain the following

$$\begin{bmatrix} A_+(\lambda) \\ B_+(\lambda) \end{bmatrix} = \begin{bmatrix} f\left(x_0^+,\lambda\right) & f\left(x_0^+,-\lambda\right) \\ f'\left(x_0^+,\lambda\right) & f'\left(x_0^+,-\lambda\right) \end{bmatrix}^{-1} B \begin{bmatrix} \psi\left(x_0^-,\lambda\right) & \varphi\left(x_0^-,-\lambda\right) \\ \psi'\left(x_0^-,\lambda\right) & \varphi'\left(x_0^-,-\lambda\right) \end{bmatrix} \begin{bmatrix} A_-(\lambda) \\ B_-(\lambda) \end{bmatrix}$$

or

$$\begin{bmatrix} A_+(\lambda) \\ B_+(\lambda) \end{bmatrix} = M \begin{bmatrix} A_-(\lambda) \\ B_-(\lambda) \end{bmatrix} \tag{6}$$

where a transfer matrix $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ satisfying the relation

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} f\left(x_0^+,\lambda\right) & f\left(x_0^+,-\lambda\right) \\ f'\left(x_0^+,\lambda\right) & f'\left(x_0^+,-\lambda\right) \end{bmatrix}^{-1} B \begin{bmatrix} \psi\left(x_0^-,\lambda\right) & \varphi\left(x_0^-,-\lambda\right) \\ \psi'\left(x_0^-,\lambda\right) & \varphi'\left(x_0^-,-\lambda\right). \end{bmatrix}$$

Here

$$M_{11} = \frac{i}{2\lambda}\{f'(x_0,-\lambda)[\alpha_1\psi(x_0,\lambda) + \alpha_2\psi'(x_0,\lambda)] - f(x_0,-\lambda)[\alpha_3\psi(x_0,\lambda) + \alpha_4\psi'(x_0,\lambda)]\}$$

$$M_{12} = \frac{i}{2\lambda}\{f'(x_0,-\lambda)[\alpha_1\varphi(x_0,\lambda) + \alpha_2\varphi'(x_0,\lambda)] - f(x_0,-\lambda)[\alpha_3\varphi(x_0,\lambda) + \alpha_4\varphi'(x_0,\lambda)]\}$$

$$M_{21} = \frac{i}{2\lambda}\{-f'(x_0,\lambda)[\alpha_1\psi(x_0,\lambda) + \alpha_2\psi'(x_0,\lambda)] + f(x_0,\lambda)[\alpha_3\psi(x_0,\lambda) + \alpha_4\psi'(x_0,\lambda)]\}$$

$$M_{22} = \frac{i}{2\lambda}\{-f'(x_0,\lambda)[\alpha_1\varphi(x_0,\lambda) + \alpha_2\varphi(x_0,\lambda)] + f(x_0,\lambda)[\alpha_3\varphi(x_0,\lambda) + \alpha_4\varphi'(x_0,\lambda)]\}$$

Now, let us consider two solutions of equation (1) which satisfies condition (5) of equation (1)

$$F(x,\lambda) = \begin{cases} A_-^+\psi(x,\lambda) + B_-^+\varphi(x,\lambda) & 0 \le x < x_0 \\ A_+^+ f(x,\lambda) + B_+^+ f(x,-\lambda) & x_0 < x < \infty \end{cases}$$

and solution which satisfies condition (2) of equation (1)

$$G(x,\lambda) = \begin{cases} A_-^-\psi(x,\lambda) + B_-^-\varphi(x,\lambda) & 0 \le x < x_0 \\ A_+^- f(x,\lambda) + B_+^- f(x,-\lambda) & x_0 < x < \infty \end{cases}$$

By using asymptotic relation

$$\lim_{x\to\infty} f(x,\pm\lambda)e^{\mp i\lambda x} = 1$$

we find that

$$A_+^+ = 1, \quad B_+^+ = 0.$$

Then, from (8) we get the following relation to find the coefficients A_-^+ and B_-^+

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_-^+ \\ B_-^+ \end{bmatrix}$$

or

$$\begin{cases} A_-^+ M_{11} + B_-^+ M_{12} = 1 \\ A_-^+ M_{21} + B_-^+ M_{22} = 0 \end{cases}$$

From that we obtain

$$A_-^+ = \frac{M_{22}}{\det M},$$

$$B_-^+ = -\frac{M_{21}}{\det M}.$$

Then for solutions $F(x,\lambda)$ and $G(x,\lambda)$ we have the following relations

$$F(x,\lambda) = \begin{cases} \frac{M_{22}}{\det M}\psi(x,\lambda) - \frac{M_{21}}{\det M}\varphi(x,\lambda), & x \to 0^+ \\ f(x,\lambda), & x \to \infty \end{cases}$$

and

$$G(x,\lambda) = \begin{cases} \varphi(x,\lambda), & x \to 0^+ \\ M_{12} f(x,\lambda) + M_{22} f(x,-\lambda), & x \to \infty. \end{cases}$$

Lemma 2. *The Wronscian of solutions $F(x,\lambda)$ and $G(x,\lambda)$ determined as follows*

$$W[F(x,\lambda), G(x,\lambda)] = \frac{M_{22}(\lambda)}{\det M}, \quad x \to 0^+$$

$$W[F(x,\lambda), G(x,\lambda)] = -2i\lambda M_{22}(\lambda), \quad x \to \infty.$$

3 Spectrum of the operator L

Let L be an operator generated by the differential expression

$$\frac{1}{\rho(x)}\left\{-\frac{d^2}{dx^2} + q(x)\right\}$$

in the space $L_2(-\infty,\infty,\rho(x))$.

By means of a general method for the kernel of the resolvent of the operator $\left(L - \lambda^2 I\right)$ we get

$$G_-(x,t,\lambda) = \begin{cases} \varphi(t,\lambda)\psi(x,\lambda) & x \le t \\ \varphi(x,\lambda)\psi(t,\lambda) & x \ge t \end{cases}$$

and

$$G_+(x,t,\lambda) = -\frac{1}{2i\lambda\beta}\begin{cases} f^+(x,\lambda) f^-(t,\lambda) & x \le t \\ f^+(t,\lambda) f^-(x,\lambda) & x \ge t \end{cases}.$$

Then we get

$$y(x,\lambda)=\int_0^\infty G(x,t,\lambda)f(t)\,dt+$$

$$+\frac{1}{M_{22}}\begin{cases}\begin{vmatrix} f(x_0,\lambda) & (\alpha_1-1)\int_0^\infty G(x_0,t,\lambda)f(t)\,dt \\ f'(x_0,\lambda) & (\alpha_3-1)\int_0^\infty G(x_0,t,\lambda)f(t)\,dt\end{vmatrix}\psi(x,\lambda) & 0\le x<x_0 \\ \begin{vmatrix} (\alpha_1-1)\int_0^\infty G(x_0,t,\lambda)f(t)\,dt & [\alpha_1\psi(x_0,\lambda)+\alpha_1\psi'(x_0,\lambda)] \\ (\alpha_3-1)\int_0^\infty G(x_0,t,\lambda)f(t)\,d & [\alpha_3\psi(x_0,\lambda)+\alpha_4\psi'(x_0,\lambda)]\end{vmatrix} f(x,\lambda) & x_0<x<\infty.\end{cases}$$

These facts and analytical properties of the $M_{22}(\lambda)$. make valid the following statement.

Theorem 2. *The eigenvalues of the problem (1-5) are the zeros of the function* $M_{22}(\lambda)$.

Theorem 3. *The spectrum of the operator consists of a continuous spectrum filling the positive half-axis* $[0,\infty)$ *on which there may exist spectral singularities coinciding with the numbers* $\left(\frac{n}{2\beta}\right)^2$.

4 Conclusion

In our study, we have examined various aspects related to spectral and scattering problems in impulsive Sturm–Liouville boundary value problems with periodic complex potentials on the semi-axis. While many studies have looked at the spectral analysis of these problems, most have focused on real potentials. Our approach stands out as we use transfer matrices to investigate eigenvalues and spectral singularities. Using this method, we have been able to identify sets of eigenvalues and spectral singularities under specific conditions.

References

Aghamaliyeva, L., Gasimov, Y., & Valdes, J.N. (2023). On a generalization of the Wirtinger inequality and some its applications. *Studia Universitatis Babe-Bolyai Mathematica, 68*(2), 237–247 DOI: 10.24193/subbmath.2023.2.01

Bairamov, E., Erdal, I., & Yardimci, S. (2018). Spectral properties of an impulsive Sturm–Liouville operator. *Journal of Inequalities and Applications*, 1-16.

Efendiev, R.F., Annaghili, S. (2023). Inverse Spectral Problem of Discontinuous Non-self-adjoint Operator Pencil with Almost Periodic Potentials. *Azerbaijan Journal of Mathematics*, *13*(1).

Efendiev, R.F., Orudzhev, H.D., & Bahlulzade, S J. (2021). Spectral analysis of the discontinuous Sturm-Liouville operator with almost-periodic potentials. *Advanced Mathematical Models Applications*, *6*(3), 266-277.

Efendiev, R.F., Orudzhev, H.D., & El-Raheem, Z.F. (2016). Spectral analysis of wave propagation on branching strings. *Boundary Value Problems,* , 1-18.

Efendiev, R., Gasimov, Y. (2022). Inverse spectral problem for pt-symmetric Schrodinger operator on the graph with loop. *Global and Stochastic Analysis*, *9*(2), 67-77.

Gasimov, Y.S. (2013). On a shape design problem for one spectral functional. *Journal of Inverse and Ill-Posed Problems, 21*(5), 629-637.

Hald, O.H. (1984). Discontinuous inverse eigenvalue problems. *Communications on Pure and Applied Mathematics*, *37*(5), 539-577.

Naimark, M.A. (1954). Investigation of the spectrum and the expansion in eigenfunctions of a nonselfadjoint operator of the second order on a semi-axis. *Trudy Moskov. Mat. Obš.*, 3, 181–270.